

§ 5.2 Bialgebras

\mathcal{C} : finite ring cat, $F: \mathcal{C} \rightarrow \text{Vec}$ fiber functor.

$$F_{x,y}: F(x) \otimes F(y) \rightarrow F(x \otimes y), \quad x, y \in \mathcal{C}.$$

$$H := \text{End}(F)$$

$$\Delta: H \rightarrow H \otimes H \quad \text{and} \quad \varepsilon: H \rightarrow k.$$

$$\alpha: \text{End}(F) \otimes \text{End}(F) \xrightarrow{\sim} \text{End}(F \otimes F) \xrightarrow{\text{isom}} \text{End}(F \otimes F) \cdot \alpha(y, \otimes y) |_{F(x) \otimes F(y)} = \eta \cdot (F_{x,y} \otimes 1)^{-1} F(y).$$

$$\Delta(a) = \alpha^{-1}(\tilde{\Delta}(a)), \quad \tilde{\Delta}(a)_{x,y} = F_{x,y}^{-1} a_{x \otimes y} F_{x,y} : F(x) \otimes F(y) \rightarrow F(x \otimes y) \\ : F(x) \otimes F(y) \xrightarrow{F_{x,y}} F(x \otimes y) \xrightarrow{a_{x \otimes y}} F(x \otimes y) \xrightarrow{F_{x,y}^{-1}} F(x) \otimes F(y)$$

$$\varepsilon(a) = a_1 \in k : F(1) \rightarrow F(1).$$

Theorem 5.2.1 Let $H = \text{End}(F)$

(i) The alg. H is a coalg with Δ and ε .

proof: (i) Coassociativity of Δ :

$$(\text{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \text{id}) \circ \Delta.$$

$$\forall a \in \text{End}(F), \quad \Delta(a) = \sum b_i \otimes c_i, \quad b_i, c_i \in \text{End}(F).$$

$$\Delta(a)_{(x,y)} := \sum_i b_i|_x \otimes c_i|_y : F(x) \otimes F(y) \rightarrow F(x) \otimes F(y).$$

$$\Delta(a) \in \text{End}(F) \otimes \text{End}(F), \quad \tilde{\Delta}(a) \in \text{End}(F \otimes F).$$

$$\tilde{\Delta}(a)_{x,y} = \Delta(a)_{(x,y)}$$

$$\cdot \text{Then } ((\Delta \otimes \text{id}) \circ \Delta(a))_{(x,y,z)} = (\sum \Delta(b_i) \otimes c_i)_{(x,y,z)}$$

$$= \sum \Delta(b_i)|_{(x,y)} \otimes c_i|_z = \sum F_{x,y}^{-1} b_i|_{x \otimes y} F_{x,y} \otimes c_i|_z.$$

$$= (F_{x,y}^{-1} \otimes \text{id}_{F(z)}) \circ (\sum b_i|_{x \otimes y} \otimes c_i|_z) \circ (F_{x,y} \otimes \text{id}_{F(z)})$$

$$= (F_{x,y}^{-1} \otimes \text{id}_{F(z)}) \circ F_{x \otimes y, z}^{-1} \circ \tilde{\Delta}(a)_{x \otimes y, z} \circ F_{x \otimes y, z} \circ (F_{x,y} \otimes \text{id}_{F(z)}) \quad (**)$$

$$((\text{id} \otimes \Delta) \circ \Delta(a))_{(x,y,z)}$$

$$= (\text{id}_{F(x)} \otimes F_{y,z}^{-1}) \circ F_{x, y \otimes z}^{-1} \circ a_{x \otimes (y \otimes z)} \circ F_{x, y \otimes z} \circ (\text{id}_{F(x)} \otimes F_{y,z}) \quad (**).$$

Let $\Phi_{x,y,z}$: associat. isom. in \mathcal{C} .

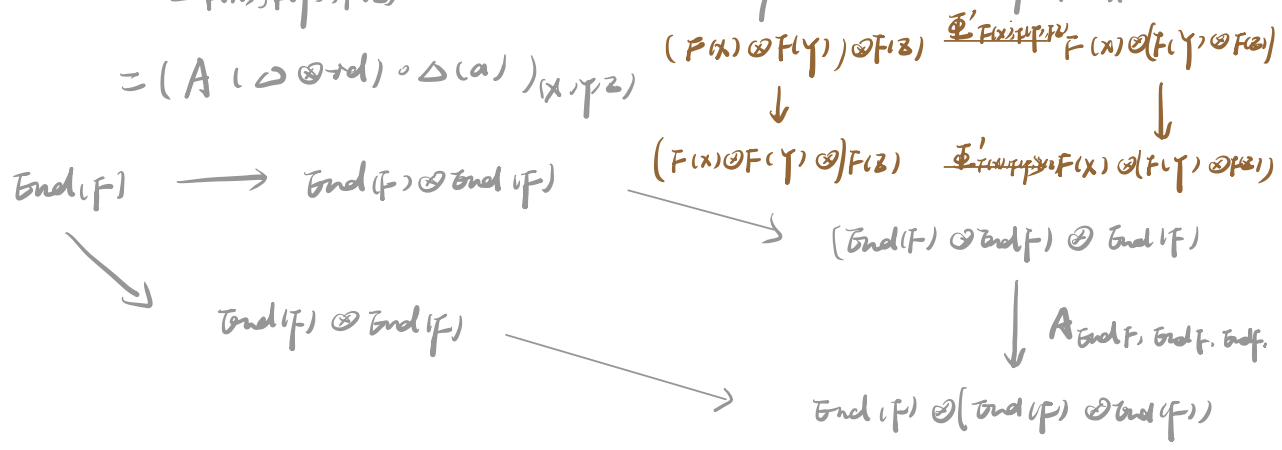
$$\text{By nat. of } a \in \text{End}(F), \Rightarrow F(\Phi_{x,y,z}) \circ a_{x \otimes (y \otimes z)} = a_{x \otimes (y \otimes z)} \circ F(\Phi_{x,y,z}).$$

$$\Rightarrow (\ast\ast) = \underbrace{(\text{id}_{F(X)} \otimes J_{Y,Z}^{-1}) \cdot J_{X,Y \otimes Z}^{-1} \circ F(\Phi_{X,Y,Z}) \cdot a_{X \otimes Y, Z}}_{\textcircled{1}} \cdot \underbrace{a_{X \otimes Y, Z} \circ F(\Phi^{-1}) \cdot J_{X,Y \otimes Z}}_{\textcircled{2}} \cdot (\text{id}_{F(X)} \otimes J_{Y,Z})$$

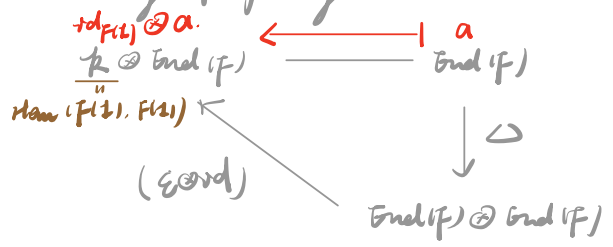
By (2.23) P30.

$$\begin{array}{ccc} (F(X) \otimes F(Y)) \otimes F(Z) & \xrightarrow{\Phi'_{F(X), F(Y), F(Z)}} & F(X) \otimes (F(Y) \otimes F(Z)) \\ \downarrow J_{X,Y} \otimes \text{id} & & \downarrow \text{id}_{F(X)} \otimes J_{Y,Z} \\ F(X \otimes Y) \otimes F(Z) & & F(X) \otimes F(Y \otimes Z) \\ \downarrow J_{X \otimes Y, Z} & & \downarrow J_{X, Y \otimes Z} \\ F(X \otimes Y) \otimes F(Z) & \xrightarrow{F(\Phi)} & F(X \otimes (Y \otimes Z)) \end{array}$$

$$\begin{aligned} \Rightarrow (\ast\ast) &= \Phi'_{F(X), F(Y), F(Z)} \circ ((\Delta \otimes \text{id}) \circ \Delta(a))_{(X,Y,Z)} \circ \Phi^{-1}_{F(X), F(Y), F(Z)} \\ &= (A((\Delta \otimes \text{id}) \circ \Delta(a)))_{(X,Y,Z)} \end{aligned}$$



Commutativity property: $(\text{id} \otimes \varepsilon) \circ \Delta = \text{id}_{\text{End}(F)} = (\varepsilon \otimes \text{id}) \circ \Delta$.

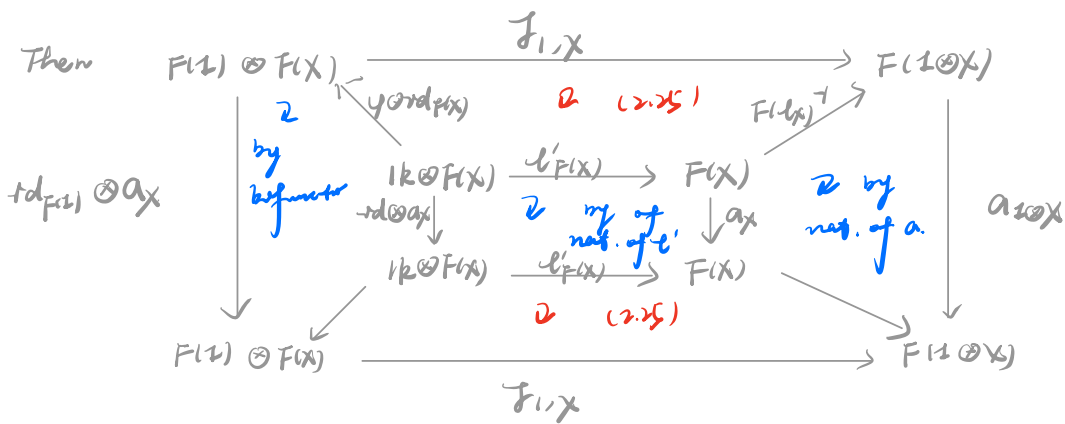


$$\begin{aligned} (\varepsilon \otimes \text{id}) \circ \Delta(a) &= \sum_j \varepsilon(b_j) \otimes c_j \\ &= \sum_j b_j|_1 \otimes c_j \\ \text{we want: } \sum_j b_j|_1 \otimes c_j &= \text{id}_{F(Z)} \otimes a. \\ \text{i.e. } \text{id}_{F(Z)} \otimes a_X &\stackrel{?}{=} \sum_j b_j|_2 \otimes c_j|_X \end{aligned}$$

By P31 (2.25)

$$\begin{array}{ccc} k \otimes F(X) & \xrightarrow{\mathcal{L}_{F(X)}} & F(X) \\ y \otimes \text{id}_{F(X)} \downarrow & \Downarrow & \downarrow F(\mathcal{L}_X)^{-1} \\ F(Z) \otimes F(X) & \xrightarrow{J_{Z,X}} & F(Z \otimes X) \end{array}$$

$$\begin{aligned} & \Delta(a)_{(1), X} \\ & \parallel \\ & \tilde{\Delta}(a)_{1, X} \\ & \parallel \\ & J_{1, X}^{-1} \circ a_{1 \otimes X} \circ J_{1, X} \end{aligned}$$



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(vi) Δ, ε are unital algebra homomorphisms.

proof: $\Delta(\text{rd}_F)_{X,Y} = \tilde{\Delta}(\text{rd}_F)_{X,Y} = \mathcal{J}_{X,Y}^{-1} \circ \text{rd}_{F(X) \otimes F(Y)} \circ \mathcal{J}_{X,Y} = \text{rd}_{F(X) \otimes F(Y)}$
 $\Delta(\text{rd}_F)_{X,Y} = \alpha^{-1}(\text{rd}_{F(X) \otimes F(Y)}) = \text{rd}_{F(X)} \otimes \text{rd}_{F(Y)} = (\text{rd}_F)|_X \otimes (\text{rd}_F)|_Y$

$$\Delta(\eta \nu) \stackrel{?}{=} \Delta(\eta) \Delta(\nu)$$

since $\tilde{\Delta}(\eta \nu)_{X,Y} = \mathcal{J}_{X,Y}^{-1} \circ \eta_{X \otimes Y} \circ \nu_{X \otimes Y} \circ \mathcal{J}_{X,Y} = \tilde{\Delta}(\eta)_{X,Y} \circ \tilde{\Delta}(\nu)_{X,Y}$

$\Rightarrow \Delta \dots$

$$\varepsilon(\text{rd}_F)$$

$$\varepsilon(\eta \nu)$$

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Th. 5.2.6. (Reconstruction theorem)

The assignments

$$(C, F) \xrightarrow{\theta} H = \text{End}(F) \quad H \xrightarrow{\xi} (\text{Rep}(H), \text{Forget})$$

are mutually inverse bijections between (1) and (2).

(1) finite ring cat. \mathcal{C} with fiber functor $F: \mathcal{C} \rightarrow \text{Vec}$,

up to tensor eq. and isomorphism classes of f.d. bialg. H over k .

(2) isomorphism classes of f.d. bialg. H over k . *如果两个条件, 那么在 (1) 中 ν 是 tensor eq. 是 must eq.*

proof: (1) $\text{End}(F)$ vs f.d.

Pro. prop. generator P of \mathcal{C} .

Consider $\text{End}(F) \xrightarrow{i} \text{Hom}_{\text{vec}}(F(P), F(P)) \quad a \mapsto a_P$

for $\forall X \in \mathcal{C}$, $\exists P^{\otimes I} \xrightarrow{\tau} X$ for some index set I .

$$\begin{array}{ccccccc}
 0 \rightarrow F(\ker \tau) & \rightarrow & F(P^{\otimes I}) & \xrightarrow{F(\tau)} & F(X) & \rightarrow & 0 \\
 \downarrow \alpha_{\ker} & & \downarrow (\alpha_P)^{\otimes I} & & \downarrow \alpha_X & & \\
 0 \rightarrow F(\ker \tau) & \rightarrow & F(P^{\otimes I}) & \xrightarrow{F(\tau)} & F(X) & \rightarrow & 0
 \end{array}$$

\downarrow by nat of α . \downarrow $\exists! \beta$

(2) mutually inverse.

$\mathcal{O}_3(H) = \mathcal{O}(\text{Rep}(H), \text{Forget}) = \text{End}(\text{Forget}) \cong H$ bialg.

$\text{End}(\text{Forget}) \xrightarrow{\quad} H$

$a_H : \text{Forg}(H) \rightarrow \text{Forg}(H)$

$\alpha \xrightarrow{\quad} \alpha_H(1_H)$

$\alpha' \xleftarrow{\beta} 1_H$

s.t. for $\forall H$ -module V ,

$\alpha'_V(V) = h.V$ α' is a nat. transf.

① $\alpha(1_H) = \alpha'_H(1) = h.1 = h$

② $\beta(\alpha) = \beta(\alpha_H(1)) = \beta(h) = \alpha'$
 $\alpha_H(1) = h'$

$$\begin{array}{ccc}
 \text{Forg}(H \otimes V) & \xrightarrow{\alpha'_V} & \text{Forg}(H \otimes V) \\
 \downarrow \text{Forg}(\phi) & & \downarrow \text{Forg}(\phi) \\
 \text{Forg}(H \otimes W) & \xrightarrow{\alpha'_W} & \text{Forg}(H \otimes W)
 \end{array}$$

$\downarrow \phi(V)$ $\downarrow h.\phi(V)$

for any $H \otimes V$, $V \in \mathcal{C}_H$, define $\phi_V : H \otimes H \rightarrow H \otimes V$

$h \xrightarrow{\quad} h.V$

$\alpha_V(V) = \alpha_V(\text{Forg}(\phi_V)(1))$

$= (\alpha_V \circ \text{Forg}(\phi_V))(1) = (\text{Forg}(\phi_V) \circ \alpha_H)(1)$

$= \text{Forg}(\phi_V)(\alpha_H(1)) = h'.V$

$\Rightarrow \alpha' = \alpha$

$$\begin{array}{ccc}
 \text{Forg}(H) & \xrightarrow{\alpha_H} & \text{Forg}(H) \\
 \downarrow \text{Forg}(\phi_V) & & \downarrow \text{Forg}(\phi_V) \\
 \text{Forg}(V) & \xrightarrow{\alpha_V} & \text{Forg}(V)
 \end{array}$$

③ α is a bialg. trans.

α is a coalg. hom.

(1) $\varepsilon_H \circ \alpha = \varepsilon_{\text{End}(\text{Forg})} \Leftrightarrow \varepsilon_H \circ \alpha(a) = \varepsilon_{\text{End}}(a) \Leftrightarrow \varepsilon_H(\alpha_H(1)) = \alpha_1$

$$\begin{array}{ccc}
 \text{Forg}(H) & \xrightarrow{\alpha_H} & \text{Forg}(H) \\
 \downarrow & & \downarrow \\
 k & \xrightarrow{\alpha_1} & k
 \end{array}$$

$$a \xrightarrow{\alpha} a_H(1_H)$$

$$(v) \Delta_H \cdot \alpha = (\alpha \otimes \alpha) \cdot \Delta_{\text{End}} \Leftrightarrow \Delta_H \circ \alpha(1_H) = (\alpha \otimes \alpha) \cdot \Delta_{\text{End}}(1_H)$$

$$\Delta_{\text{End}}(a)_{(x,y)} = \delta(a)_{x,y} = a_{x \otimes y}$$

$$\Delta_H(a_H(1_H)) = a_{H \otimes H}(1 \otimes 1)$$

$$= (\alpha \otimes \alpha) \Delta_{\text{End}}(1_H)$$

$$\begin{array}{ccc} \text{Forg}(H) & \xrightarrow{\alpha_H} & \text{Forg}(H) \\ \text{Forg}(\Delta) \downarrow & \searrow & \downarrow \text{Forg}(\Delta) \\ \text{Forg}(H \otimes H) & \xrightarrow{\alpha_{H \otimes H}} & \text{Forg}(H \otimes H) \end{array}$$

$$\Delta_{\text{End}}(a) = \sum a_{i_1} \otimes a_{i_2}$$

$$(\alpha \otimes \alpha)(1_H) = a_{H \otimes H}(1) \otimes a_{H \otimes H}(1)$$

$$(\Delta_{\text{End}}(a)_{(H,H)})(1 \otimes 1)$$

$$\cdot \exists \theta((C, F)) = \zeta(\text{End}(F)) = (\text{Rep}(\text{End}(F)), \text{Forget}) \cong (C, F)$$

$$\text{Let } \tilde{F}: C \rightarrow \text{Rep}(\text{End}(F))$$

$$x \mapsto (F(x), P_{F(x)})$$

$$\text{Hom}_C(x, y) \ni f \mapsto F(f) \in \text{Hom}_{\text{Rep}(\text{End}(F))}^{(F(x), F(y))}$$

$$F(x) \in \text{Vec}, \text{End}(F) \xrightarrow{P_{F(x)}} \text{Hom}_k(F(x), F(x))$$

$$\text{id}_F \mapsto \text{id}_{F(x)}$$

$$a \mapsto a_x$$

$$\Rightarrow P_{F(x)} \text{ is a rep.}$$

① \tilde{F} is a tensor functor

(1) faithful, exact.

$$(v) \tilde{F}(x) \otimes \tilde{F}(y) \cong \tilde{F}(x \otimes y)$$

$$\left(\tilde{F}(1_C) \cong k \text{ by } F(1_C) = k \right)$$

$$\downarrow \text{i.e. } (F(x), P_{F(x)}) \otimes (F(y), P_{F(y)}) \cong (F(x \otimes y), P_{F(x \otimes y)})$$

$$\cong (F(x) \otimes F(y), P_{F(x)} \otimes P_{F(y)})$$

$$\forall a \in \text{End}(F), P_{F(x) \otimes F(y)}(a) = (P_{F(x)} \otimes P_{F(y)})(a) = P_{F(x)}(a_{i_1}) \otimes P_{F(y)}(a_{i_2})$$

$$= a_{i_1}|_x \otimes a_{i_2}|_y = J_{x,y}^{-1} \cdot \frac{a_{x \otimes y}}{P_{F(x \otimes y)}(a)} \cdot J_{x,y}$$

$$P_{F(x \otimes y)}(a) = a_{x \otimes y}$$

(2) The monoidal structure. Given by \tilde{F} is a fiber functor

③ equi.

Pr. Cor. 1.8.11 $F \cong \text{Hom}_{\mathcal{C}}(\mathcal{V}, -)$ for $\mathcal{V} \in \mathcal{C}$.

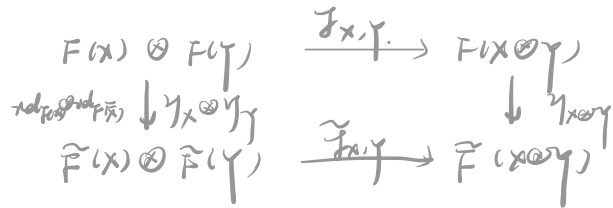
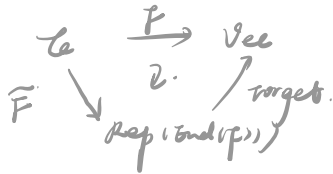
$\Rightarrow F \cong \text{Hom}_{\mathcal{C}}(P, -)$, P is proj. gen. of \mathcal{C} .

$\Rightarrow \text{Hom}_{\mathcal{C}}(P, -) : \mathcal{C} \rightarrow \text{Rep}(\text{End}(P))$

$\Rightarrow \tilde{F}$ is an eq.

$\Rightarrow \text{Rep}(\text{End}(P)) \cong \mathcal{C}$.

• forget $\cong F$ as tensor functor.



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Left twist

$$\begin{aligned}
 (f \otimes 1)(\Delta \otimes \text{id})(f) &\in H \otimes H \otimes H, & H^f &= (H, m, \mu, \Delta^f, \varepsilon) \\
 &= (1 \otimes f) \cdot (\text{id} \otimes \Delta)(f) & \Delta^f(h) &= f \cdot \Delta(h) \cdot f^{-1}
 \end{aligned}$$

f is invertible $f \in H \otimes H$

Proposition. 设 H 是 f.d. bialg, $\tau \in H \otimes H$ 是 (左) twist.

记 $F: \text{Rep}(H) \rightarrow \text{Vec}$ 和 $F': \text{Rep}(H^\tau) \rightarrow \text{Vec}$ 是忘却 (fiber) 函子.

若张量范畴等价 $(E, \tau): \text{Rep}(H) \rightarrow \text{Rep}(H^\tau)$ 满足:

$F' \circ E$ 与 F 作为张量函子自然同构, 则 $H^\tau \cong H$ as bialgs.

Remark. (E, τ) 的定义如下:

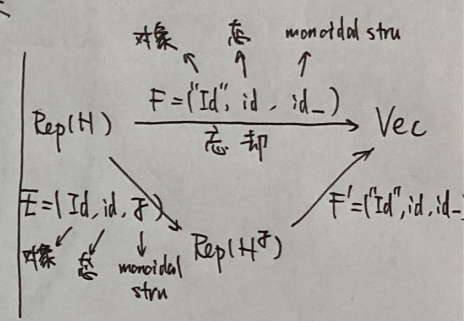
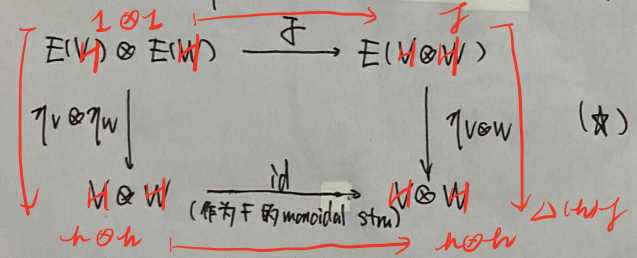
$$E: V \mapsto E(V) \text{ (作为 } H\text{-mod)}, \quad \begin{array}{ccc} V & & V \\ f \downarrow & \mapsto & f \downarrow \\ W & & W \end{array}$$

monoidal stru $\tau: E(V) \otimes E(W) \xrightarrow{\cong} E(V \otimes W), v \otimes w \mapsto \tau(v \otimes w)$.

由左 twist 的定义可知 (左乘) τ 是 $\text{Rep}(H^\tau)$ 中的 morphism.

Proof. 设自然同构 $\eta \in \text{Nat}(F' \circ E, F)$ 满足

$\forall V, W \in \text{Rep}(H), \text{Vec}$ 中的下图交换:



事实上, 由于自然的线性同构 η_V 与所有 H -mod map 交换,

即 $\forall f \downarrow_W$ in $\text{Rep}(H), \eta_V \downarrow \eta_W$ 故 \exists 可逆元 $h \in H, \text{ s.t. } \forall V \in \text{Rep}(H), \eta_V: v \mapsto h \cdot v$ (h 与 V 无关).

(理由: 先判断 $V=W=H$ 以及自由模的情形, 而对一般的 H -mods, 考虑 V 和 W 的 free resolution $\cong \eta$ 的自然性)

因此, 对 $1 \otimes 1 \in E(V) \otimes E(W)$ 应用 (a) 的交换性, 可得

$$\Delta(h) \tau = h \otimes h, \quad \text{即 } \tau = \Delta(h^{-1})(h \otimes h).$$

这说明: 在 twists τ 和 $1 \otimes 1$ 在 [EGNO15, P115 第三段] 的意义下等价, 从而 $H^\tau \cong H^{1 \otimes 1} = H$ (as bialgs).

留到 §5.14 解决吧 ~